Harmonic and monogenic functions in superspace

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Abstract. The aim of this work is to further extend the analytic theory of monogenic functions in superspace and to construct a set of operators that allow to construct an explicit basis for the spaces of symplectic spherical harmonics.

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1. Introduction

In a previous set of papers (see a.o. [7, 8, 9, 11]) we have developed a theory of harmonic analysis and Clifford analysis in superspace. Superspaces are spaces equipped with both a set of commuting variables and a set of anti-commuting variables (generating a so-called Grassmann algebra). They are usually studied from the point of view of algebraic or differential geometry (see [2, 14]). Our approach, on the other hand, was based on a generalization of harmonic and Clifford analysis by introducing differential operators such as a Dirac and Laplace operator.

The null-solutions of the super Dirac operator ([7]) are called super monogenic functions. Their properties are similar to those of the classical bosonic monogenic functions. In [8] spherical monogenics were used to obtain a Fischer decomposition when the super dimension is not even and negative. In [4] a Cauchy integral theorem and Morera's theorem was proven for super monogenic functions. In this paper we continue the development of monogenic function theory in superspace with Liouville's theorem, a maximum modulus theorem and the Taylor expansion for monogenic functions.

In the second part of this paper we consider a generalization of the Kelvin transformation to the purely fermionic case. We cannot generalize the transform directly, but we obtain a set of operators that exhibit the same behaviour. These operators yield a recursive procedure to prove statements concerning fermionic harmonics. They also allow to construct, in principle, a basis for the space of purely fermionic spherical harmonics. Such a basis is necessary to construct e.g. a Mehler formula in Grassmann algebras (see [5]). Moreover, the construction then immediately yields a basis for the super spherical harmonics using the results obtained in [11].

2. Preliminaries

Superspaces are spaces where one considers not only commuting (bosonic) but also anti-commuting (fermionic) co-ordinates (see a.o. [2, 14]). In our approach to superspace (see [7]), we start with the real algebra $\mathcal{P} \otimes \mathcal{C} = \mathrm{Alg}(x_i, \grave{x}_j) \otimes \mathrm{Alg}(e_i, \grave{e}_j) = \mathrm{Alg}(x_i, e_i; \grave{x}_j, \grave{e}_j)$, $i = 1, \ldots, m$, $j = 1, \ldots, 2n$ generated by

- m commuting variables x_i and m orthogonal Clifford generators e_i
- $2n$ anti-commuting variables $\grave{x}_i$ and $2n$ symplectic Clifford generators $\grave{e}_i$

subject to the multiplication relations

$$\begin{cases} x_i x_j = x_j x_i \\ \grave{x}_i \grave{x}_j = -\grave{x}_j \grave{x}_i \\ x_i \grave{x}_j = \grave{x}_j x_i \end{cases} \quad \text{and} \quad \begin{cases} e_j e_k + e_k e_j = -2\delta_{jk} \\ \grave{e}_{2j} \grave{e}_{2k} - \grave{e}_{2k} \grave{e}_{2j} = 0 \\ \grave{e}_{2j-1} \grave{e}_{2k-1} - \grave{e}_{2k-1} \grave{e}_{2j-1} = 0 \\ \grave{e}_{2j-1} \grave{e}_{2k} - \grave{e}_{2k} \grave{e}_{2j-1} = \delta_{jk} \\ e_j \grave{e}_k + \grave{e}_k e_j = 0 \end{cases}$$

and where moreover all elements e_i, $\grave{e}_j$ commute with all elements x_i, $\grave{x}_j$. The algebra generated by all generators e_i, $\grave{e}_j$ is denoted by $\mathcal{C}$. In the case where $n = 0$ we have that $\mathcal{C} \cong \mathbb{R}_{0,m}$, the standard orthogonal Clifford algebra with signature $(-1, \ldots, -1)$. When $m = 0$, we have that $\mathcal{P} \otimes \mathcal{C} = \Lambda_{2n} \otimes \mathcal{W}_{2n}$, with Λ_{2n} the Grassmann algebra generated by the $\grave{x}_i$ and $\mathcal{W}_{2n}$ the Weyl algebra generated by the $\grave{e}_j$. The most important element of the algebra $\mathcal{P} \otimes \mathcal{C}$ is the vector variable $x = \underline{x} + \underline{\grave{x}}$ with

$$\underline{x} = \sum_{i=1}^{m} x_i e_i \ , \qquad \underline{\grave{x}} = \sum_{j=1}^{2n} \grave{x}_j \grave{e}_j .$$

The square of x is scalar-valued and equals $x^2 = \sum_{j=1}^{n} \grave{x}_{2j-1} \grave{x}_{2j} - \sum_{j=1}^{m} x_j^2 = \underline{\grave{x}}^2 + \underline{x}^2$. The bosonic part $\underline{x}^2$ is invariant under $SO(m)$ while $\underline{\grave{x}}^2$ is invariant under the symplectic group $Sp(2n)$, so x^2 is invariant under $SO(m) \times Sp(2n)$.

On the other hand, the super Dirac operator is defined as

$$\partial_x = \partial_{\underline{\grave{x}}} - \partial_{\underline{x}} = 2 \sum_{j=1}^{n} \left(\grave{e}_{2j} \partial_{\grave{x}_{2j-1}} - \grave{e}_{2j-1} \partial_{\grave{x}_{2j}} \right) - \sum_{j=1}^{m} e_j \partial_{x_j} .$$

Its square is the super Laplace operator $\Delta = \partial_x^2 = 4 \sum_{j=1}^{n} \partial_{\grave{x}_{2j-1}} \partial_{\grave{x}_{2j}} - \sum_{j=1}^{m} \partial_{x_j}^2 = \Delta_f + \Delta_b$. Also, if we let ∂_x act on x, we obtain $\partial_x x = m - 2n = M = (x \partial_x)$ where M is the so-called super-dimension. Note that the anti-commuting variables behave as if their dimension is negative. The numerical parameter M gives a global characterization of our superspace.

Furthermore we introduce the super Euler operator by

$$\mathbb{E} = \mathbb{E}_b + \mathbb{E}_f \quad = \quad \sum_{j=1}^{m} x_j \partial_{x_j} + \sum_{j=1}^{2n} \grave{x}_j \partial_{\grave{x}_j}.$$

The operators Δ, x^2 and $\mathbb{E} + M/2$ satisfy (see [7, 9])

$$\begin{aligned}\left[\Delta/2, x^2/2\right] &= \mathbb{E} + M/2\\ \left[\Delta/2, \mathbb{E} + M/2\right] &= 2\Delta/2\\ \left[x^2/2, \mathbb{E} + M/2\right] &= -2x^2/2\end{aligned}$$

which are the canonical commutation relations of $\mathfrak{sl}_2$, see e.g. [15]. Similar to classical Clifford analysis ∂_x and x generate the $\mathfrak{osp}(1|2)$ Lie superalgebra. This means we have the same computation rules as in Clifford analysis with substitution of the Euclidean dimension by the super dimension.

The super Euler operator allows us to decompose $\mathcal{P}$ as

$$\mathcal{P} \quad = \quad \bigoplus_{k=0}^{\infty} \mathcal{P}_k, \quad \mathcal{P}_k = \{p \in \mathcal{P} \mid \mathbb{E}p = kp\}.$$

Now we define spherical harmonics in superspace.

Definition 2.1. An element $F \in \mathcal{P}$ is a spherical harmonic of degree k if it satisfies

$$\begin{aligned}\Delta F &= 0\\ \mathbb{E}F &= kF, \quad \text{i.e. } F \in \mathcal{P}_k.\end{aligned}$$

Moreover the space of all spherical harmonics of degree k is denoted by $\mathcal{H}_k$.

In the purely bosonic case we denote $\mathcal{H}_k$ by $\mathcal{H}_k^b$, in the purely fermionic case by $\mathcal{H}_k^f$. The $\mathfrak{sl}$-commutation relations lead to the following decomposition (see [8]).

Lemma 2.2 (Fischer decomposition 1). *If $M \notin -2\mathbb{N}$, $\mathcal{P}$ decomposes as*

$$\mathcal{P} = \bigoplus_{k=0}^{\infty} \mathcal{P}_k = \bigoplus_{j=0}^{\infty} \bigoplus_{k=0}^{\infty} x^{2j} \mathcal{H}_k. \tag{2.1}$$

If $m = 0$, then the decomposition is given by

$$\Lambda_{2n} = \bigoplus_{k=0}^{n} \left(\bigoplus_{j=0}^{n-k} \underline{x}^{2j} \mathcal{H}_k^f \right). \tag{2.2}$$

In the same way we define the space of spherical monogenics of degree k as $\mathcal{M}_k = \ker \partial_x \cap \mathcal{P}_k \otimes \mathcal{C}$. It is clear that every spherical monogenic is a spherical harmonic. This allows us to refine the Fischer decomposition, leading to (see [8])

Lemma 2.3 (Fischer decomposition 2). *If $M \notin -2\mathbb{N}$, $\mathcal{P}_k \otimes \mathcal{C}$ decomposes as*

$$\mathcal{P}_k \otimes \mathcal{C} = \bigoplus_{i=0}^{k} x^i \mathcal{M}_{k-i}.$$

If $m = 0$, then the decomposition is given by

$$\mathcal{P}_k \otimes \mathcal{W}_{2n} = \bigoplus_{i=0}^{k} \underline{x}^{\grave{}\, i} \mathcal{M}^f_{k-i}, \quad k \leq n \tag{2.3}$$

$$\mathcal{P}_{2n-k} \otimes \mathcal{W}_{2n} = \bigoplus_{i=0}^{k} \underline{x}^{\grave{}\, 2n-2k+i} \mathcal{M}^f_{k-i}, \quad k \leq n. \tag{2.4}$$

For the purely fermionic case we thus obtain the full decomposition

$$\Lambda_{2n} \otimes \mathcal{W}_{2n} = \bigoplus_{k=0}^{n} \bigoplus_{j=0}^{2n-2k} \underline{x}^{\grave{}\, j} \mathcal{M}^f_k. \tag{2.5}$$

In general a function $f \in C^1(\Omega) \otimes \Lambda_{2n}$ with Ω an open domain in $\mathbb{R}^m$ is called left-monogenic (or also monogenic or super monogenic) in Ω if $\partial_x f = 0$.

In [11] the space $\mathcal{H}_k$ was decomposed into irreducible pieces under the action of the group $SO(m) \times Sp(2n)$.

Theorem 2.4 (Decomposition of $\mathcal{H}_k$). *Under the action of $SO(m) \times Sp(2n)$ the space $\mathcal{H}_k$ decomposes as*

$$\mathcal{H}_k = \bigoplus_{i=0}^{\min(n,k)} \mathcal{H}^b_{k-i} \otimes \mathcal{H}^f_i \oplus \bigoplus_{j=0}^{\min(n,k-1)-1} \bigoplus_{l=1}^{\min(n-j, \lfloor \frac{k-j}{2} \rfloor)} f_{l,k-2l-j,j} \mathcal{H}^b_{k-2l-j} \otimes \mathcal{H}^f_j, \tag{2.6}$$

with $f_{k,p,q} = \sum_{s=0}^{k} \binom{k}{s} \frac{(n-q-s)!}{\Gamma(\frac{m}{2}+p+k-s)} \underline{x}^{2k-2s} \underline{x}^{\grave{}\, 2s}$.

Finally we will need the Cauchy-Kowalewskaia extension in superspace. We start from the space

$$\mathcal{A}_k = \mathcal{P}_k \otimes \mathcal{C} \bmod x_1 \mathcal{P}_{k-1} \otimes \mathcal{C},$$

by which we mean the elements of $\mathcal{P}_k \otimes \mathcal{C}$ that do not contain x_1. Let $f \in \mathcal{A}_k$, then the Cauchy-Kowalewskaia extension is defined as

$$CK(f) = \sum_{i=0}^{k} \frac{x_1^i}{i!} (-e_1 \widetilde{\partial_x})^i f$$

with $\widetilde{\partial_x} = \partial_x + e_1 \partial_{x_1}$.We also define a map $R : \mathcal{M}_k \longrightarrow \mathcal{A}_k$ by

$$R(g) = g \bmod x_1 \mathcal{P}_{k-1} \otimes \mathcal{C}.$$

Now, as was proven in [12],

Theorem 2.5. *CK is an isomorphism (of right $\mathcal{C}$-modules) between $\mathcal{A}_k$ and $\mathcal{M}_k$, with inverse R.*

3. Monogenic functions theory in superspace

In this section we study analytic properties of super monogenic functions. The classical analogues in bosonic Clifford analysis can be found in [3, 13]. In [4] we already obtained Morera's theorem and Cauchy integral formulas for monogenic functions in superspace.

In this section we will often express super functions using the basis provided by the fermionic Fischer decomposition (2.5). Consider a basis for the fermionic spherical monogenics $\{M_k^l\} \subset \mathcal{M}_k^f$ as a right $\mathcal{W}_{2n}$-module. We normalize this basis by $M_k^{l,j} = c_{k,j} M_k^l$, such that $\partial_{\underline{x}} \underline{x}^j M_k^{l,j} = \underline{x}^{j-1} M_k^{l,j-1}$. We can then expand a general superfunction f as

$$f(x) \quad = \quad \sum_{k=0}^{n} \sum_{j=0}^{2n-2k} \sum_{l=1}^{\dim \mathcal{M}_k^f} f_{j,k,l}(\underline{x})\, \underline{x}^j M_k^{l,j}. \tag{3.1}$$

Expressing that f is supermonogenic on Ω leads to

$$0 \quad = \quad \sum_{k=0}^{n} \sum_{j=1}^{2n-2k} \sum_{l=1}^{\dim \mathcal{M}_k^f} f_{j,k,l}(\underline{x})\, \underline{x}^{j-1} M_k^{l,j} - \sum_{k=0}^{n} \sum_{j=0}^{2n-2k} \sum_{l=1}^{\dim \mathcal{M}_k^f} (\partial_{\underline{x}} f_{j,k,l}(\underline{x}))\, \underline{x}^j M_k^{l,j},$$

with $f_{j,k,l} : \Omega \subset \mathbb{R}^m \to \mathbb{R}_{0,m}$. So we find

$$\partial_{\underline{x}} f_{2n-2k,k,l} = 0 \quad \text{and} \quad \partial_{\underline{x}} f_{j,k,l} = f_{j+1,k,l} \text{ for } j < 2n-2k. \tag{3.2}$$

This implies that all the bosonic parts are polyharmonic and thus analytic. So all super monogenic functions on Ω are analytic functions on Ω.

In Clifford analysis Liouville's theorem holds (see [13]), which states that when a monogenic function satisfies $|f(\underline{x})| \leq C(1 + |\underline{x}|)^k$ then f is a polynomial of degree k, for $k \geq 0$. $|f(\underline{x})|$ is defined as the norm of $f(\underline{x})$ as an element of the 2^m-dimensional vector space $\mathbb{R}_{0,m}$ for every $\underline{x} \in \mathbb{R}^m$. We first prove a slight generalization of this result.

Lemma 3.1. *Let $f : \mathbb{R}^m \to \mathbb{R}_{0,m}$ be a function satisfying $\partial_{\underline{x}} f = g$ in $\mathbb{R}^m$ with $g \in \mathbb{R}[x_1, \ldots, x_m] \otimes \mathbb{R}_{0,m}$. If there exists a constant C and a natural number k such that in $\mathbb{R}^m$*

$$|f(\underline{x})| \quad \leq \quad C(1 + |\underline{x}|)^k$$

then f is a polynomial of degree k.

Proof. Let l be the degree of the polynomial g. Then by the surjectivity of the Dirac operator, there exists a polynomial h of degree $l+1$ for which $\partial_{\underline{x}} h = g$. This means that $f - h$ is monogenic and there is a constant D for which

$$\begin{aligned} |f(\underline{x}) - h(\underline{x})| &\leq C(1 + |\underline{x}|)^k + D(1 + |\underline{x}|)^{l+1} \\ &\leq (C + D)(1 + |\underline{x}|)^{\max(k,l+1)}. \end{aligned}$$

This means that using Liouville's theorem, $f - h$ is polynomial of degree $\max(k, l+1)$. So because h is a polynomial of degree $l+1$, f is a polynomial of degree

$\max(k, l+1)$. Now because $|f(\underline{x})| \leq C(1+|\underline{x}|)^k$, f cannot be a polynomial of degree higher than k. $\square$

Now we show that Liouville's theorem also holds for super monogenic functions. There are several different generalizations of Liouville's theorem to superspace possible, we will prove two versions. We begin with a definition. Recall that f decomposes as

$$f(x) = \sum_{k=0}^{n} \sum_{j=0}^{2n-2k} \sum_{l=1}^{\dim \mathcal{M}_k^f} f_{j,k,l}(\underline{x})\, \underline{x}^j M_k^{l,j}.$$

Then we say that $|f(x)|_1 \leq C(1+|\underline{x}|)^p$ if and only if $|f_{j,k,l}(\underline{x})| \leq C(1+|\underline{x}|)^{p-k-j}$ for all j, k and l.

Theorem 3.2. *(Liouville)*
Let f be a superfunction monogenic in $\mathbb{R}^m$. If there exists $C \in \mathbb{R}$ and $p \in \mathbb{N}$ such that

$$|f(x)|_1 \leq C(1+|\underline{x}|)^p$$

in $\mathbb{R}^m$, then f is a super polynomial of degree p.

Proof. We expand f as in (3.1) and because f is monogenic it satisfies the equations in (3.2). From this and the classical Liouville's theorem we find that $f_{2n-2k,k,l}$ is a polynomial of degree $p-2n+k$. The rest of the lemma is proved using induction and lemma 3.1. Indeed, because $f_{j+1,k,l}$ is a polynomial and because $\partial_{\underline{x}} f_{j,k,l} = f_{j+1,k,l}$, we have that $f_{j,k,l}$ is a polynomial of degree $p-j-k$. $\square$

We could also consider the super monogenic function $f(x)$ as a bosonic function $f(\underline{x}) : \Omega \subset \mathbb{R}^m \rightarrow \Lambda_{2n} \otimes \mathcal{C}$. Then we define that $|f(\underline{x})|_2 \leq C(1+|\underline{x}|)^p$ means that with the same expansion as before $|f_{j,k,l}(\underline{x})| \leq C(1+|\underline{x}|)^p$ must hold for every j, k and l. This is equivalent to defining $|f(\underline{x})|_2$ as the norm of $f(\underline{x})$ in the vector space $\Lambda_{2n} \otimes \mathcal{C}$. With exactly the same techniques as in theorem 3.2 we find

Theorem 3.3. *(Liouville II)*
Let f be a superfunction monogenic in $\mathbb{R}^m$. If there exists $C \in \mathbb{R}$ and $p \in \mathbb{N}$ such that

$$|f(\underline{x})|_2 \leq C(1+|\underline{x}|)^p$$

in $\mathbb{R}^m$, then f is a super polynomial of degree $p+n$.

To find a non trivial maximum modulus theorem, the second interpretation of $|f(\underline{x})|$ is necessary.

Theorem 3.4. *(Maximum Modulus)*
Let f be a monogenic superfunction in the open and connected set Ω. If there exists a point $\underline{a} \in \Omega$ such that

$$|f(\underline{x})|_2 \leq |f(\underline{a})|_2, \quad \textit{i.e. } |f_{j,k,l}(\underline{x})| \leq |f_{j,k,l}(\underline{a})| \quad \forall j,k,l$$

for all $\underline{x} \in \Omega$, then f is a (monogenic) element of $\Lambda_{2n} \otimes \mathcal{C}$.

Proof. We start from the maximum modulus theorem for bosonic harmonic functions, which states that if for for a harmonic function $g(\underline{x})$, $|g(\underline{x})| \leq |g(\underline{a})|$ holds for all $\underline{x} \in \Omega$, then g is a constant function in Ω. In particular we find that the harmonic $f_{2n-2k,k,l}$ and $f_{2n-2k-1,k,l}$ are constants. Because $f_{2n-2k,k,l} = \partial_{\underline{x}} f_{2n-2k-1,k,l}$ we even find that $f_{2n-2k,k,l}$ is zero. This can be continued using induction, if $f_{j,k,l}$ is a constant, then $f_{j-1,k,l}$ is harmonic and therefore a constant and $f_{j,k,l}$ is zero. So we finally find that $f_{0,k,l}$ is a constant and the other $f_{j,k,l}$ are zero. □

Now we consider a general monogenic function f on the open ball with radius R, $\mathbb{B}^m(R)$, which we expand as in (3.1). Because every $f_{j,k,l}$ is analytic, there exists for each of them an open neighbourhood of the origin $\Omega_{j,k,l}$, where it can be written as a Taylor series

$$f_{j,k,l}(\underline{x}) \quad = \quad \sum_{t=0}^{\infty} (P_t f_{j,k,l})(\underline{x}) \tag{3.3}$$

with $P_t f_{j,k,l}$ a polynomial of degree t. The series and all its derived series converge normally in $\Omega_{j,k,l}$. We define the open neighbourhood of the origin $\Omega = \cap_{j,k,l} \Omega_{j,k,l}$. (There are only finitely many (j,k,l).)

Lemma 3.5. *With the notations introduced above the following holds in Ω for every $k \leq n$ and l,*

$$\partial_{\underline{x}} P_t f_{2n-2k,k,l} = 0 \quad \text{and} \quad \partial_{\underline{x}} P_{t+1} f_{j,k,l} = P_t f_{j+1,k,l} \text{ for } j < 2n-2k.$$

Proof. The fact that $\partial_{\underline{x}} P_t f_{2n-2k,k,l} = 0$ is exactly lemma 11.3.3 in [3]. The other cases can be proven using the same technique. For every $\underline{\alpha}$ with $|\underline{\alpha}| = t$,

$$\begin{aligned} \partial_{\underline{x}}^{\underline{\alpha}} \partial_{\underline{x}} P_{t+1} f_{j,k,l} &= \left(\partial_{\underline{x}}^{\underline{\alpha}} \partial_{\underline{x}} f_{j,k,l} \right)(0) \\ &= \left(\partial_{\underline{x}}^{\underline{\alpha}} f_{j+1,k,l} \right)(0) \\ &= \partial_{\underline{x}}^{\underline{\alpha}} P_t f_{j+1,k,l}, \end{aligned}$$

which proves the lemma. □

Now we define

$$P_s f = \sum_{j,k,l}^{j+k \leq s} P_{s-j-k} f_{j,k,l} \, \underline{x}\grave{}^j M_k^{l,j}.$$

Lemma 3.5 implies $\partial_x P_s f = 0$, so $P_s f \in \mathcal{M}_s$ and

$$f = \sum_{s=0}^{\infty} P_s f \tag{3.4}$$

in Ω. This is a vectorial summation of the series in (3.3).

For $\alpha = (\alpha_2, \cdots, \alpha_m) \in \mathbb{N}^{m-1}$, $\beta \in \{0,1\}^{2n}$ and $|\alpha| + |\beta| = \sum \alpha_i + \sum \beta_i = k$ we define $\alpha! = (\alpha_2!) \cdots (\alpha_m!)$ and

$$x^{\alpha,\beta} = x_2^{\alpha_2} \ldots x_m^{\alpha_m} x\grave{}_1^{\beta_1} \ldots x\grave{}_{2n}^{\beta_{2n}}. \tag{3.5}$$

Theorem 3.6. *For f left monogenic in $\mathbb{B}^m(R)$, there exists an open neighbourhood of the origin Ω, for which*

$$f = \sum_{s=0}^{\infty} \sum_{|\alpha|+|\beta|=s} \frac{1}{\alpha!} CK(x^{\alpha,\beta})[\partial_{x^{\alpha,\beta}} f](0)$$

where $\partial_{x^{\alpha,\beta}} = \partial_{x_2}^{\alpha_2} \dots \partial_{x_m}^{\alpha_m} \partial_{\grave{x}_{2n}}^{\beta_{2n}} \dots \partial_{\grave{x}_1}^{\beta_1}$.

Proof. Because of theorem 2.5 we know that

$$CK(P_s f \mod x_1 \mathcal{P}_{s-1} \otimes \mathcal{C}) = P_s f.$$

We also have that for every superpolynomial R_i of degree i

$$\sum_{\alpha,\beta;|\alpha|+|\beta|=i} \frac{1}{\alpha!} x^{\alpha,\beta} \partial_{x^{\alpha,\beta}} R_i \equiv R_i \mod x_0 \mathcal{P}_{i-1} \otimes \mathcal{C},$$

yielding

$$\begin{aligned} P_s f &= \sum_{|\alpha|+|\beta|=s} \frac{1}{\alpha!} CK(x^{\alpha,\beta}) \partial_{x^{\alpha,\beta}} P_s f \\ &= \sum_{|\alpha|+|\beta|=s} \frac{1}{\alpha!} CK(x^{\alpha,\beta})[\partial_{x^{\alpha,\beta}} f](0). \end{aligned}$$

The lemma then follows from equation (3.4). □

From this theorem we find that the set $CK(x^{\alpha,\beta})$ takes over the role played by the powers of the complex variable z when expanding holomorphic functions in the complex plane into Taylor series.

Remark 3.7. In the bosonic case one can prove that the domain of convergence Ω is actually equal to the original $\mathbb{B}^m(R)$, see [3]. The proof there can also be done in superspace. When $M \notin -2\mathbb{N}$ the fundamental solution of the Dirac operator ([10]) can be expanded similarly to the bosonic case (see [6]). The Cauchy integral theorem for monogenic functions (see [4]) then leads to a Taylor expansion for the monogenic function in the entire ball. Because of unicity this has to equal the Taylor expansion in formula (3.4).

4. Basis for the space of symplectic harmonics

Theorem 2.4 implies that bases for $\mathcal{H}_j^b$ and $\mathcal{H}_l^f$ suffice to obtain a basis for $\mathcal{H}_k$. Bases for $\mathcal{H}_j^b$ are well known (e.g. [1]), here we will construct a set of operators that allow to generate all elements of $\mathcal{H}_l^f$. In this section we will always consider the purely fermionic case, so $m = 0$ and $\mathcal{P} = \Lambda_{2n}$.

We first introduce a symplectic transformation on the variables. The transformation $\tilde{\cdot} : \Lambda_{2n} \to \Lambda_{2n}$ is a linear transformation which acts on elements $(a, b \in \Lambda_{2n})$ as

$$\widetilde{ab} = \tilde{b}\tilde{a} \quad \text{and} \quad \widetilde{\grave{x}_{2i-1}} = \grave{x}_{2i} \quad , \quad \widetilde{\grave{x}_{2i}} = -\grave{x}_{2i-1}.$$

In this notation we find $\underline{\grave{x}}^2 = \frac{1}{2}\sum_{j=1}^{2n} \grave{x}_j \widetilde{\grave{x}_j} = \widetilde{\underline{\grave{x}}^2}$.

One way to construct a basis for the bosonic harmonic polynomials is the Kelvin transformation (see [1]), also known as Maxwell's representation theorem ([16]). As the Kelvin transformation itself does not generalize to the fermionic case, we introduce operators that have the same effect. Recall that the Kelvin transformation of a (bosonic) function $u(\underline{x})$ in $\mathbb{R}^m$ is given by

$$K[u] \quad = \quad |\underline{x}|^{2-m} u(\frac{\underline{x}}{|\underline{x}|^2}).$$

This transformation is clearly an involution ($K^2 = 1$). It can be proven that $K[\partial_{\underline{x}}^{\underline{\alpha}} |\underline{x}|^{2-m}]$ is a harmonic polynomial of degree $j = |\underline{\alpha}|$. We will now deform the results from [1] in a way that will be useful. Inspired by those results we will construct a similar method for the fermionic case. Using the involution property we can write for a certain bosonic spherical harmonic H_j^b

$$\partial_{\underline{x}}^{\underline{\alpha}} |\underline{x}|^{2-m} \quad = \quad K[H_j^b] = |\underline{x}|^{2-m-2j} H_j^b.$$

We can now add one more partial derivative, ∂_{x_k}

$$\partial_{x_k} \partial_{\underline{x}}^{\underline{\alpha}} |\underline{x}|^{2-m} \quad = \quad |\underline{x}|^{2-m-2j-2}(|\underline{x}|^2 \partial_{x_k} + (2-m-2j)x_k) H_j^b$$

and see that using the fact that the above is homogeneous of degree $2-m-j-1$, for a certain spherical harmonic H_{j+1}^b

$$\begin{aligned} H_{j+1}^b &= K[\partial_{x_k} \partial_{\underline{x}}^{\underline{\alpha}} |\underline{x}|^{2-m}] \\ &= (|\underline{x}|^2 \partial_{x_k} + (2-m-2j)x_k) H_j^b \\ &= (|\underline{x}|^2 \partial_{x_k} + (2-m-2(\mathbb{E}_b - 1))x_k) H_j^b \\ &= (|\underline{x}|^2 \partial_{x_k} - (2\mathbb{E}_b + m - 4)) H_j^b. \end{aligned}$$

Now we can do something similar in the fermionic case. Using the $\mathfrak{sl}_2$-commutation relations we find

$$\begin{aligned} [\Delta_f, (\mathbb{E}_f - n - 2)\grave{x}_k] &= 2\Delta_f \grave{x}_k + \mathbb{E}_f[\Delta_f, \grave{x}_k] - (n+2)[\Delta_f, \grave{x}_k] \\ &= 2\grave{x}_k \Delta_f - (4\mathbb{E}_f - 4n)\partial_{\tilde{\grave{x}}_k}. \end{aligned}$$

We also used

$$\begin{aligned} [\Delta_f, \grave{x}_{2j-1}] &= 4\partial_{\grave{x}_{2j-1}} \partial_{\grave{x}_{2j}} \grave{x}_{2j-1} - \grave{x}_{2j-1} 4\partial_{\grave{x}_{2j-1}} \partial_{\grave{x}_{2j}} \\ &= -4\partial_{\grave{x}_{2j}} \end{aligned}$$

which is similar for $\grave{x}_{2j}$, so $[\Delta_f, \grave{x}_k] = -4\partial_{\tilde{\grave{x}}_k}$. Again using the $\mathfrak{sl}_2$-commutation relations we obtain

$$[\Delta_f, \underline{\grave{x}}^2 \partial_{\tilde{\grave{x}}_k}] = (4\mathbb{E}_f - 4n)\partial_{\tilde{\grave{x}}_k},$$

finally yielding

$$[\Delta_f, (\mathbb{E}_f - n - 2)\grave{x}_k + \underline{\grave{x}}^2 \partial_{\tilde{\grave{x}}_k}] = 2\grave{x}_k \Delta_f. \tag{4.1}$$

Considering some facts about the operator we created here we can find a basis for $\mathcal{H}_k^f$. The first steps are similar to the steps to form a basis for the bosonic harmonic polynomials via the Kelvin transformation ([1]).

Definition 4.1. The operator $D_k : \Lambda_{2n} \to \Lambda_{2n}$ is given by

$$D_k \quad = \quad (\mathbb{E}_f - n - 2)\grave{x}_k + \underline{\grave{x}}^2 \partial_{\tilde{\grave{x}}_k}, \quad k = 1, \ldots, 2n.$$

Using (4.1) we find that for a spherical harmonic $H_l \in \mathcal{H}_l^f$, $\Delta_f D_k H_l = 0$, so $D_k H_l \in \mathcal{H}_{l+1}^f$. Hence we have obtained

Lemma 4.2. *The operators from definition* 4.1 *map* $\mathcal{H}_l^f$ *into* $\mathcal{H}_{l+1}^f$*:*

$$D_k \mathcal{H}_l^f \quad \subset \quad \mathcal{H}_{l+1}^f, \quad \forall l \leq n-1.$$

We show that the operators D_j are anticommutative.

Lemma 4.3. *The operators in definition* 4.1 *generate a Grassmann algebra isomorphic to* Λ_{2n}*, i.e.*

$$\{D_j, D_k\} = 0, \quad \forall j, k \in \{1, \cdots, 2n\}.$$

Proof. The property follows from the calculations

$$\underline{\grave{x}}^2 \partial_{\tilde{\grave{x}}_j} \underline{\grave{x}}^2 \partial_{\tilde{\grave{x}}_k} + \underline{\grave{x}}^2 \partial_{\tilde{\grave{x}}_k} \underline{\grave{x}}^2 \partial_{\tilde{\grave{x}}_j} \quad = \quad -\underline{\grave{x}}^2 \left(\grave{x}_j \partial_{\tilde{\grave{x}}_k} + \grave{x}_k \partial_{\tilde{\grave{x}}_j} \right)$$

and

$$\begin{aligned} &(\mathbb{E}_f - n - 2)\, \grave{x}_j \underline{\grave{x}}^2 \partial_{\tilde{\grave{x}}_k} + \underline{\grave{x}}^2 \partial_{\tilde{\grave{x}}_k} (\mathbb{E}_f - n - 2)\, \grave{x}_j \\ &= \underline{\grave{x}}^2 (\mathbb{E}_f - n)\, \grave{x}_j \partial_{\tilde{\grave{x}}_k} + \underline{\grave{x}}^2 (\mathbb{E}_f - n - 1)\, \partial_{\tilde{\grave{x}}_k} \grave{x}_j \\ &= \underline{\grave{x}}^2 \grave{x}_j \partial_{\tilde{\grave{x}}_k} + \underline{\grave{x}}^2 (\mathbb{E}_f - n - 1) \left(\partial_{\tilde{\grave{x}}_k} \grave{x}_j \right). \end{aligned}$$

So we find that

$$\{D_j, D_k\} \quad = \quad \underline{\grave{x}}^2 (\mathbb{E}_f - n - 1) \left((\partial_{\tilde{\grave{x}}_k} \grave{x}_j) + (\partial_{\tilde{\grave{x}}_j} \grave{x}_k) \right).$$

The only case where the above is non trivial is when $\tilde{\grave{x}}_k = \pm \grave{x}_j$. This however implies that $\grave{x}_k = \mp \tilde{\grave{x}}_j$. $\square$

As the operators generate a Grassmann algebra, we define the symplectic $\underline{D}^2$ as $\sum_{j=1}^n D_{2j-1} D_{2j}$. Because $D_{2j-1} D_{2j} 1 = (n+1)(n \grave{x}_{2j-1} \grave{x}_{2j} - \underline{\grave{x}}^2)$ we find the important formula

$$\underline{D}^2 1 \quad = \quad 0. \tag{4.2}$$

Because of the isomorphism with the Grassmann algebra we can define fermionic polynomials in the operators D_k. This can be used to prove the surjectivity of these operators.

Lemma 4.4. *For all* $j \leq n$ *and for every* $p \in \mathcal{P}_j$ *there exists a* $q \in \mathcal{P}_{j-2}$ *such that*

$$p(\underline{D}) 1 = c_j (p(\underline{\grave{x}}) + \underline{\grave{x}}^2 q(\underline{\grave{x}}))$$

with c_j *some non zero constant, only depending on* j*.*

Proof. Because of linearity it only has to be proven for monomials

$$\underline{D}^{\underline{\alpha}} = D_1^{\alpha_1} \dots D_{2n}^{\alpha_{2n}} \quad \alpha_j \in \{0,1\}.$$

Because the operators D_j generate a Grassmann algebra we find that the order of the operators is unimportant, up to the sign. The case $j = 1$ is trivial. Using induction we assume the lemma holds for j, and calculate for $|\underline{\alpha}| = j$,

$$\begin{aligned} D_k \underline{D}^{\underline{\alpha}} 1 &= c_j (j-n-1) \grave{x}_k [\underline{\grave{x}}^{\underline{\alpha}} + \underline{\grave{x}}^2 q] + \underline{\grave{x}}^2 \partial_{\grave{x}_k} \underline{D}^{\underline{\alpha}} 1 && (4.3)\\ &= c_j (j-n-1) \left(\grave{x}_k \underline{\grave{x}}^{\underline{\alpha}} + \underline{\grave{x}}^2 q' \right). && (4.4) \end{aligned}$$

So we find that $c_{j+1} = (j-n-1)c_j$ and the proposed formula still holds for $j+1$, this concludes the proof. □

Corollary 4.5. *For a* $p \in \mathcal{P}_j$, $p(\underline{D})1 = 0$ *if and only if* $p(\underline{\grave{x}}) = \underline{\grave{x}}^2 q(\underline{\grave{x}})$ *for some* $q \in \mathcal{P}_{j-2}$.

Proof. The if part follows from formula (4.2) and the only if part from lemma 4.4. □

Now we can prove that the set of operators $\{D_k\}$ is surjective for $\mathcal{H}_l^f \rightarrow \mathcal{H}_{l+1}^f$.

Proposition 4.6. *The operators from definition* 4.1 *act surjective in the sense that* $\bigoplus_{k=1}^{2n} D_k \mathcal{H}_l^f = \mathcal{H}_{l+1}^f$ *and*

$$span \, \{\underline{D}^{\underline{\alpha}} 1, |\underline{\alpha}| = l\} = \mathcal{H}_l^f.$$

Proof. For every $p(\underline{\grave{x}}) \in \mathcal{H}_l^f$ we have by lemma 4.2 that $p(\underline{D})1 \in \mathcal{H}_l^f$. Combining this with lemma 4.4 and the uniqueness of the Fischer decomposition (2.2) (see [9]) we find that $p(\underline{D})1 = c_l p(\underline{\grave{x}})$, so $\mathcal{H}_l^f \subset span\{\underline{D}^{\underline{\alpha}} 1, |\underline{\alpha}| = l\} \subset \mathcal{H}_l^f$. □

It is now possible to use proposition 4.6 to construct very explicitly a basis for the space of fermionic harmonics $\mathcal{H}_k^f$. As this construction is very tedious and complicated (requiring careful relabelings of the variables), we choose to omit the proof from this paper (although we present a brief sketch below). In some sense, proposition 4.6 is all one needs in order to prove theorems concerning fermionic harmonics (see e.g. [5]), as it yields a recursive procedure to reduce statements concerning harmonics of degree k to degree $k-1$. Moreover, the set of operators D_j are important as they capture the essence of the classical Kelvin transformation in a set of differential operators which allow for more straightforward generalizations.

Sketch of construction:

Proposition 4.6 combined with corollary 4.5 implies that a basis for Λ_{2n}^k mod $\underline{\grave{x}}^2 \Lambda_{2n}^{k-2}$ is mapped to a basis for $\mathcal{H}_k^f$ under the morphism $p(\underline{\grave{x}}) \rightarrow p(D)1$. An important tool in the construction of an explicit basis for $\mathcal{H}_k^f$ is therefore the following lemma.

Lemma 4.7. *For a Grassmann algebra Λ_{2n} generated by the $\grave{x}_j$, the following relation holds,*

$$\grave{x}_1\grave{x}_2\ \grave{x}_3\grave{x}_4\cdots\grave{x}_{2j-1}\grave{x}_{2j} \equiv \frac{(-1)^j}{j!}(\grave{x}_{2j+1}\grave{x}_{2j+2}+\cdots+\grave{x}_{2n-1}\grave{x}_{2n})^j \ \textit{mod}\ \underline{\grave{x}}^2.$$

It can now be proven that a basis for Λ_{2n}^k mod $\underline{\grave{x}}^2\Lambda_{2n}^{k-2}$ is given by the monomials in Λ_{2n}^k excluding the sets

$$\grave{x}_{i_1}\grave{x}_{i_2}\cdots\grave{x}_{i_{k-2j}}S^j_{n-k+2j}(\grave{y}_1,\grave{y}_2,\cdots,\grave{y}_{2n-2k+4j}),$$

with $\grave{x}_{i_l}\neq\widetilde{\grave{x}_{i_t}}$, $1\leq l,t\leq k-2j$ and $\{\grave{y}_1,\ldots,\grave{y}_{2n-2k+4j}\}$ a relabeling of $\{\grave{x}_1,\ldots,\grave{x}_{2n}\}\setminus\{\grave{x}_{i_1},\ldots,\grave{x}_{i_{k-2j}},\widetilde{\grave{x}}_{i_1},\cdots,\widetilde{\grave{x}}_{i_{k-2j}}\}$ for $1\leq j\leq\lfloor k/2\rfloor$.

In this notation, the set S^j_p is defined as follows.

Definition 4.8. The set of monomials $S_p^j(\grave{y}_1,\grave{y}_2,\cdots,\grave{y}_{2p})$ is the subset of the set

$$(\grave{y}_1\grave{y}_2)^{\beta_1}(\grave{y}_3\grave{y}_4)^{\beta_2}\cdots(\grave{y}_{2p-1}\grave{y}_{2p})^{\beta_p},\quad \beta_i\in\{0,1\},\ \sum_i\beta_i=j$$

where the powers satisfy

$$\beta_p+\beta_{p-1}+\cdots+\beta_{p-2j+2} \quad=\quad j,$$

or where for some $t\in\{2,3,\cdots,j-1\}$ and some $i_l, l=1,2,\cdots,t-1$ with $i_l<i_{l+1}$ and $i_l<p-2j+2l$

$$\beta_{i_1}\beta_{i_2}\cdots\beta_{i_{t-1}}(\beta_p+\beta_{p-1}+\cdots+\beta_{p-2j+2t}) \quad=\quad j-t+1,$$

or where

$$\beta_{i_1}\beta_{i_2}\cdots\beta_{i_{j-1}}\beta_{i_{j-1}+1} \quad=\quad 1,$$

for some $i_l, l=1,2,\cdots,j-1$ with $i_l<i_{l+1}$ and $i_l<p-2j+2l$.

Lemma 4.7 and definition 4.8 allow to obtain the following theorem after a laborious procedure which we omit.

Theorem 4.9. *A basis for the space $\mathcal{H}_k$ is given by the set $\{\underline{D}^{\underline{\alpha}}1 \,|\, |\underline{\alpha}|=k\}$ excluding the sets*

$$D_{i_1}D_{i_2}\cdots D_{i_{k-2j}}S^j_{n-k+2j}(D_{s_1},D_{s_2},\cdots,D_{s_{2n-2k+4j}})1$$

with $D_{i_l}\neq\widetilde{D}_{i_t}$, $1\leq l,t\leq k-2j$ and $D_{s_1},D_{s_2},\cdots,D_{s_{2n-2k+4j}}$ the operators from $(D_1,\cdots,D_{2n})$ which are not in $(D_{i_1},D_{i_2},\cdots,D_{i_{k-2j}})$ and $(\widetilde{D}_{i_1},\widetilde{D}_{i_2},\cdots,\widetilde{D}_{i_{k-2j}})$ in unchanged order, for $1\leq j\leq\lfloor k/2\rfloor$.

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